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Nodal and Multiple Constant Sign Solution for Equations with the p-Laplacian

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Dedicated to Arrigo Cellina and James Yorke

Abstract. We consider nonlinear elliptic equations driven by the p-Laplacian with a nonsmooth potential (hemivariational inequalities). We obtain the existence of multiple nontrivial solutions and we determine their sign (one positive, one negative and the third nodal). Our approach uses nonsmooth critical point theory coupled with the method of upper-lower solutions.

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1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary ∂Z. We consider the following nonlinear elliptic problem with nonsmooth potential (hemivariational inequality):

$$\left\{ \begin{array}{l} -\text{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) \in \partial j\big(z, x(z)\big) \text{ a.e. on } Z\,, \\ x|_{\partial Z} = 0, \;\; 1 < p < \infty\,. \end{array} \right\} \tag{1.1}$$

Here $j(z,x)$ is a measurable function on $Z \times \mathbb{R}$ and $x \to j(z,x)$ is locally Lipschitz and in general nonsmooth. By $\partial j(z,\cdot)$ we denote the generalized subdifferential of $j(z,\cdot)$ in the sense of Clarke [3]. The aim of this lecture is to produce multiple nontrivial solutions for problem (1.1) and also determine their sign (positive, negative or nodal (sign-changing) solutions). Recently this problem was studied for equations driven by the p-Laplacian with a C^1-potential function (single-valued

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right hand side), by Ambrosetti-Garcia Azorero-Peral Alonso [1], Carl-Perera [2], Garcia Azorero-Peral Alonso [7], Garcia Azorero-Manfredi-Peral Alonso [8], Zhang-Chen-Li [15] and Zhang-Li [16]. In [1], [7], [8], the authors consider certain nonlinear eigenvalue problems and obtain the existence of two strictly positive solutions for all small values of the parameter $\lambda \in \mathbb{R}$ (i.e., for all $\lambda \in (0, \lambda^*)$). In [2], [15], [16] the emphasis is on the existence of nodal (sign changing) solutions. Carl-Perera [2] extend to the p-Laplacian the method of Dancer-Du [6], by assuming the existence of an ordered pair of upper-lower solutions. In contrast, Zhang-Chen-Li [15] and Zhang-Li [16], base their approach on the invariance properties of certain carefully constructed pseudogradient flow. Our approach here is closer to that of Dancer-Du [6] and Carl-Perera [2], but in contrast to them, we do not assume the existence of upper-lower solutions, but instead we construct them and we use a recent alternative variational characterization of the second eigenvalue λ_2 of $(-\triangle_p, W_0^{1,p}(Z))$ due to Cuesta-de Figueiredo-Gossez [5], together with a nonsmooth version of the second deformation theorem due to Corvellec [4].

2. Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle , \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz. The generalized directional derivative $\varphi^0(x; h)$ of φ at $x \in X$ in the direction $h \in X$, is given by

$$\varphi^0(x; h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda} .$$

The function $h \to \varphi^0(x; h)$ is sublinear continuous and so it is the support function of a nonempty, convex and w^*-compact set $\partial\varphi(x) \subseteq X^*$ defined by

$$\partial\varphi(x) = \left\{ x^* \in X^* : \langle x^*, h \rangle \leq \varphi^0(x; h) \quad \text{for all } \ h \in X \right\} .$$

The multifunction $x \to \partial\varphi(x)$ is known as the generalized subdifferential or subdifferential in the sense of Clarke. If φ is continuous convex, then $\partial\varphi(x)$ coincides with the subdifferential in the sense of convex analysis. If $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$. We say that $x \in X$ is a critical point of φ, if $0 \in \partial\varphi(x)$. The main reference for this subdifferential, is the book of Clarke [3].

Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, we say that φ satisfies the nonsmooth Palais-Smale condition at level $c \in \mathbb{R}$ (the nonsmooth PS_c-condition for short), if every sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $\varphi(x_n) \to c$ and $m(x_n) = \inf\{\|x\| : x^* \in \partial\varphi(x_n)\} \to 0$ as $n \to \infty$, has a strongly convergent subsequence. If this is true at every level $c \in \mathbb{R}$, then we say that φ satisfies the PS-condition.

Definition 2.1. *Let Y be a Hausdorff topological space and E_0, E, D nonempty, closed subsets of Y with $E_0 \subseteq E$. We say that $\{E_0, E\}$ is linking with D in Y, if the following hold:*

(a) $E_0 \cap D = \emptyset$;

(b) *for any* $\gamma \in C(E,Y)$ *such that* $\gamma|_{E_0} = id|_{E_0}$, *we have* $\gamma(E) \cap D \neq \emptyset$.

Using this geometric notion, we can have the following minimax characterization of critical values for nonsmooth, locally Lipschitz functions (see Gasinski-Papageorgiou [9], p.139).

Theorem 2.2. *If* X *is a Banach space,* E_0, E, D *are nonempty, closed subsets of* X, $\{E_0, E\}$ *are linking with* D *in* X, $\varphi : X \to \mathbb{R}$ *is locally Lipschitz,* $\sup\limits_{E_0} < \inf\limits_{D} \varphi$, $\Gamma = \{\gamma \in C(E,X) : \gamma|_{E_0} = id|_{E_0}\}$, $c = \inf\limits_{\gamma\in\Gamma} \sup\limits_{v\in E} \varphi(\gamma(v))$ *and* φ *satisfies the nonsmooth* PS_c*-condition, then* $c \geq \inf\limits_{D} \varphi$ *and* c *is a critical value of* φ.

Remark 2.3. By appropriate choices of the linking sets $\{E_0, E, D\}$, from Theorem 2.2, we obtain nonsmooth versions of the mountain pass theorem, saddle point theorem, and generalized mountain pass theorem. For details, see Gasinksi-Papageorgiou [9].

Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, we set

$$\varphi^{\cdot c} = \{x \in X : \varphi(x) < c\} \quad \text{(the strict sublevel set of } \varphi \text{ at } c \in \mathbb{R})$$

and $K_c = \{x \in X : 0 \in \partial\varphi(x), \varphi(x) = c\}$ (the critical points of φ at the level c).

The next theorem is a nonsmooth version, of the so-called "second deformation theorem" (see Gasinski-Papageorgiou [10], p.628) and it is due to Corvellec [4].

Theorem 2.4. *If* X *is a Banach space,* $\varphi : X \to \mathbb{R}$ *is locally Lipschitz, it satisfies the nonsmooth* PS*-condition,* $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$, φ *has no critical points in* $\varphi^{-1}(a,b)$ *and* K_a *is discrete nonempty and contains only local minimizers of* φ, *then there exists a deformation* $h : [0,1] \times \varphi^{\cdot b} \to \varphi^{\cdot b}$ *such that*

(a) $h(t,\cdot)|_{K_a} = Id|_{K_a}$ *for all* $t \in [0,1]$;
(b) $h(t, \varphi^{\cdot b}) \subseteq \varphi^{\cdot a} \cup K_a$;
(c) $\varphi(h(t,x)) \leq \varphi(x)$ *for all* $(t,x) \in [0,1] \times \varphi^{\cdot b}$.

Remark 2.5. In particular then $\varphi^{\cdot b} \cup K_a$ is a weak deformation retract of $\varphi^{\cdot b}$.

Let us mention a few basic things about the spectrum of $(-\triangle_p, W_0^{1,p}(Z))$, which we will need in the sequel. So let $m \in L^\infty(Z)_+$, $m \neq 0$ and consider the following weighted eigenvalue problem:

$$\left\{ \begin{array}{l} -\text{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) = \lambda m(z)|x(z)|^{p-2}x(z) \text{ a.e. on } Z\,, \\ x|_{\partial Z} = 0, \;\; 1 < p < \infty\,. \end{array} \right\} \tag{2.1}$$

Problem (2.1) has at least eigenvalue $\widehat{\lambda}_1(m) > 0$, which is simple, isolated and admits the following variational characterization in terms of the Rayleigh quotient:

$$\widehat{\lambda}_1(m) = \min\left[\frac{\|Dx\|_p^p}{\int_Z m|x|^p dz} : x \in W_0^{1,p}(Z)\,, \quad x \neq 0\right] \tag{2.2}$$

The minimum is attained on the corresponding one dimensional eigenspace $E(\lambda_1)$. By u_1 we denote the normalized eigenfunction, i.e., $\int_Z m|u_1|^p dz = 1$

(if $m \equiv 1$, then $\|u_1\|_p = 1$). We have $E(\lambda_1) = \mathbb{R}u_1$ and $u_1 \in C_0^1(\overline{Z})$ (nonlinear regularity theory, see Lieberman [13] and Gasinski-Papageorgiou [10], p.738). We set

$$C_+ = \left\{x \in C_0^1(\overline{Z}) : x(z) \geq 0 \text{ for all } z \in \overline{Z}\right\}$$

$$\text{and } \operatorname{int}C_+ = \left\{x \in C_+ : x(z) > 0 \text{ for all } z \in Z \text{ and } \frac{\partial x}{\partial n}(z) < 0 \text{ for all } z \in \partial Z\right\}.$$

The nonlinear strong maximum principle of Vazquez [14], implies that $u_1 \in \operatorname{int}C_+$.

Since $\widehat{\lambda}_1(m)$ is isolated, we can define the second eigenvalue of $(-\triangle_p, W_0^{1,p}(Z), m)$ by

$$\widehat{\lambda}_2^*(m) = \inf\left[\widehat{\lambda} : \widehat{\lambda} \text{ is an eigenvalue of (2.1), } \widehat{\lambda} \neq \widehat{\lambda}_1(m)\right] > \widehat{\lambda}_1(m)\,.$$

Also by virtue of the Liusternik-Schnirelmann theory, we can find an increasing sequence of eigenvalues $\{\widehat{\lambda}_k(m)\}_{k\geq 1}$ such that $\widehat{\lambda}_k(m) \to \infty$. These are the so-called LS-eigenvalues. We have

$$\widehat{\lambda_2}^*(m) = \widehat{\lambda}_2(m)\,,$$

i.e., the second eigenvalue and the second LS-eigenvalue coincide. The eigenvalues $\widehat{\lambda}_1(m)$ and $\widehat{\lambda}_2(m)$ exhibit the following monotonicity properties with respect to the weight function $m \in L^\infty(Z)_+$:

- If $m_1(z) \leq m_2(z)$ a.e. on Z, and $m_1 \neq m_2$, then $\lambda_1(m_2) < \lambda_1(m_1)$ (see (2.2)).
- If $m_1(z) < m_2(z)$ a.e. on Z, then $\lambda_2(m_2) < \lambda_2(m_1)$.

If $m \equiv 1$, then we write $\widehat{\lambda}_1(1) = \lambda_1$ and $\widehat{\lambda}_2(1) = \lambda_2$. Recently Cuesta-de Figueiredo-Gossez [5], produced the following alternative variational characterization of λ_2 :

$$\lambda_2 = \inf_{\gamma_0\in\Gamma_0} \sup\nolimits_{x\in\gamma_0([-1,1])} \|Dx\|_p^p \tag{2.3}$$

with $\Gamma_0 = \{\gamma_0 \in C([-1,1], S) : \gamma_0(-1) = -u_1, \gamma_0(1) = u_1\}$, $S = W_0^{1,p}(Z) \cap \partial B_1^{L^p(Z)}$ and $\partial B_1^{L^p(Z)} = \{x \in L^p(Z) : \|x\|_p = 1\}$.

Finally we recall the notions of upper and lower solution for problem (1.1).

Definition 2.6.

(a) *A function* $\overline{x} \in W^{1,p}(Z)$ *is an upper solution of* (1.1), *if* $\overline{x}|_{\partial Z} \geq 0$ *and*

$$\int_Z \|D\overline{x}\|^{p-2}(D\overline{x}, Dv)_{\mathbb{R}^N}dz \geq \int_Z uvdz$$

for all $v \in W_0^{1,p}(Z)$, $v \geq 0$ *and all* $u \in L^\eta(Z)$, $u(t) \in \partial j(t, \overline{x}(z))$ *a.e. on* Z *for some* $1 < \eta < p^*$.

(b) *A function* $\underline{x} \in W^{1,p}(Z)$ *is a lower solution of* (1.1), *if* $\overline{x}|_{\partial Z} \leq 0$ *and*

$$\int_Z \|D\underline{x}\|^{p-2}(D\underline{x}, Dv)_{\mathbb{R}^N}dz \leq \int_Z uvdz$$

for all $v \in W_0^{1,p}(Z)$, $v \geq 0$ *and all* $u \in L^\eta(Z)$, $u(z) \in \partial j(z, \underline{x}(z))$ *a.e. on* Z *for some* $1 < \eta < p^*$.

3. Multiple constant sign solutions

In this section, we produce multiple solutions of constant sign. Our approach is based on variational techniques, coupled with the method of upper lower solutions. We need the following hypotheses on the nonsmooth potential $j(z,x)$.

$\underline{H(j)_1}$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(t,0) = 0$ and $\partial j(z,0) = \{0\}$ a.e. on Z, and

(i) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;

(iii) for a.a. $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$, we have

$$|u| \leq a(z) + c|x|^{p-1} \text{ with } a \in L^\infty(Z)_+,\ c > 0\,;$$

(iv) there exists $\theta \in L^\infty(Z)_+$, $\theta(z) \leq \lambda_1$ a.e. on Z, $\theta \neq \lambda_1$ such that

$$\limsup_{|x|\to\infty} \frac{u}{|x|^{p-2}x} \leq \theta(z)$$

uniformly for a.a. $z \in Z$ and all $u \in \partial j(z,x)$;

(v) there exists $\eta, \widehat{\eta} \in L^\infty(Z)_+$, $\lambda_1 \leq \eta(z) \leq \widehat{\eta}(z)$ a.e. on Z, $\lambda_1 \neq \eta$ such that

$$\eta(z) \leq \liminf_{x\to 0} \frac{u}{|x|^{p-2}x} \leq \limsup_{x\to 0} \frac{u}{|x|^{p-2}x} \leq \widehat{\eta}(z)$$

uniformly for a.a. $z \in Z$ and all $u \in \partial j(z,x)$;

(vi) for a.a. $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$, we have $ux \geq 0$ (sign condition).

Let $\varepsilon > 0$ and $\gamma_\varepsilon \in L^\infty(Z)_+$, $\gamma_\varepsilon \neq 0$ and consider the following auxiliary problem:

$$\left\{ \begin{array}{l} -\mathrm{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) = \big(\theta(z)+\varepsilon\big)|x(z)|^{p-2}x(z) + \gamma_\varepsilon(z) \text{ a.e. on } Z\,, \\ x|_{\partial Z} = 0\,. \end{array} \right\} \quad (3.1)$$

In what follows by $\langle \cdot,\cdot \rangle$ we denote the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,p'}(Z))$ $(\frac{1}{p} + \frac{1}{p'} = 1)$. Let $A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ be the nonlinear operator defined by

$$\big\langle A(x), y \big\rangle = \int_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbb{R}^N} dz \text{ for all } x, y \in W_0^{1,p}(Z)\,.$$

We can check that A is monotone, continuous, hence maximal monotone. In particular then we can deduce that A is pseudomonotone and of type $(S)_+$.

Also let $N_\varepsilon : L^p(Z) \to L^{p'}(Z)$ be the bounded, continuous map defined by

$$N_\varepsilon(x)(\cdot) = \big(\theta(\cdot) + \varepsilon\big)|x(\cdot)|^{p-2}x(\cdot)\,.$$

Evidently due to the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we have that $N_\varepsilon|_{W_0^{1,p}(Z)}$ is completely continuous. Hence $x \to A(x) - N_\varepsilon(x)$ is pseudomonotone. Moreover, from the hypothesis on θ (see $H(j)_1(iv)$), we can show that there exists $\xi_0 > 0$ such that

$$\|Dx\|_p^p - \int_Z \theta |x|^p dz \geq \xi_0 \|Dx\|_p^p \quad \text{for all } \ x \in W_0^{1,p}(Z)\,. \tag{3.2}$$

Therefore for $\varepsilon > 0$ small the pseudomonotone operator $x \to A(x) - N_\varepsilon(x)$ is coercive. But a pseudomonotone coercive operator is surjective (see Gasinski-Papageorgiou [10], p.336). Combining this fact with the nonlinear strong maximum principle, we are led to the following existence result concerning problem (3.1).

Proposition 3.1. *If $\theta \in L^\infty(Z)_+$ is as in hypothesis $H(j)_1(iv)$, then for $\varepsilon > 0$ small problem* (3.1) *has a solution $\overline{x} \in intC_+$.*

Because of hypothesis $H(j)_1(iv)$, we deduce easily the following fact:

Proposition 3.2. *If hypotheses $H(j)_1 \to (iv)$ hold and $\varepsilon > 0$ is small, then the solution $\overline{x} \in intC_+$ obtained in Proposition* 3.1 *is a strict upper solution for* (1.1) *(strict means that $\overline{x}$ is an upper solution which is not a solution).*

Clearly $\underline{x} \equiv 0$ is a lower solution for (1.1).

Let $C = [0, \overline{x}] = \{x \in W_0^{1,p}(Z) : 0 \leq x(z) \leq \overline{x}(z) \ \text{ a.e. on } \ Z\}$. We introduce the truncation function $\tau_+ : \mathbb{R} \to \mathbb{R}$ defined by

$$\tau_+(x) = \begin{cases} 0 & \text{if } \ x \leq 0 \\ x & \text{if } \ x > 0 \end{cases} .$$

We set $j_1(z, x) = j(z, \tau_+(x))$. This is still a locally Lipschitz integrand. We introduce $\varphi_+ : W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$\varphi_+(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z j_+\big(z, x(z)\big) dz \quad \text{for all } x \in W_0^{1,p}(Z)\,.$$

The function φ_+ is Lipschitz continuous on bounded sets, hence locally Lipschitz. Using hypothesis $H(j)_1(iv)$ and (3.2), we can show that φ_+ is coercive. Moreover, due to the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, φ_+ is weakly lower semicontinuous. Therefore by virtue of Weierstrass theorem, we can find $x_0 \in C$ such that

$$\varphi_+(x_0) = \inf_C \varphi_+\,. \tag{3.3}$$

Hypothesis $H(j)_1(v)$ implies that for $\mu > 0$ small we have $\varphi_+(\mu u_1) < 0 = \varphi_+(0)$. Since $\mu u_1 \in C$, it follows that $x_0 \neq 0$. Moreover, from (3.3) we have

$$0 \leq \big\langle A(x_0), y - x_0 \big\rangle - \int_Z u_0(z)(y - x_0)(z) dz \quad \text{for all } \ y \in C\,, \tag{3.4}$$

with $u_0 \in L^{p'}(Z)$, $u_0(z) \in \partial j_+(z, x_0(z)) = \partial j(z, x_0(z))$ a.e. on Z. For $h \in W_0^{1,p}(Z)$ and $\varepsilon > 0$, we define

$$y(z) = \begin{cases} 0 & \text{if } z \in \{x_0 + \varepsilon h \le 0\} \\ x_0(z) + \varepsilon h(z) & \text{if } z \in \{0 < x_0 + \varepsilon h \le \overline{x}\} \\ \overline{x}(z) & \text{if } z \in \{\overline{x} \le x_0 + \varepsilon h\} \end{cases} .$$

Evidently $y \in C$ and so we can use it as a test function in (3.4). Then we obtain

$$0 \le \langle A(x_0) - u_0, h \rangle . \tag{3.5}$$

Because $h \in W_0^{1,p}(Z)$ was arbitrary, from (3.5) we conclude that

$$A(x_0) = u_0 \quad \Rightarrow x_0 \in W_0^{1,p}(Z) \ \text{ is a solution of (1.1)} . \tag{3.6}$$

Nonlinear regularity theory implies that $x_0 \in C_0^1(\overline{Z})$, while the nonlinear strong maximum principle of Vazquez [14], tell us that $x_0 \in \mathrm{int}C_+$.

Using the comparison principles of Guedda-Veron [11], we can show that

$$\overline{x} - x_0 \in \mathrm{int}C_+ .$$

Therefore x_0 is a local $C_0^1(\overline{Z})$-minimizer of φ, hence x_0 is a local $W_0^{1,p}(Z)$-minimizer of φ (see Gasinski-Papageorgiou [9], pp.655–656 and Kyritsi-Papageorgiou [12]). Therefore we can state the following result:

Proposition 3.3. *If hypotheses $H(j)_1$ hold, then there exists $x_0 \in C$ which is a local minimizer of φ_+ and of φ.*

If instead of (3.1), we consider the following auxiliary problem

$$\left\{ \begin{array}{l} -\mathrm{div}\big(\|Dv(z)\|^{p-2} Dv(z)\big) = \big(\theta(z) + \varepsilon\big)|v(z)|^{p-2} v(z) - \gamma_\varepsilon(z) \text{ a.e. on } Z , \\ v|_{\partial Z} = 0 . \end{array} \right\} \tag{3.7}$$

then we obtain as before a solution $\underline{v} \in -\mathrm{int}C_+$ of (3.7). We can check that this $\underline{v} \in -\mathrm{int}C_+$ is a strict lower solution for problem (1.1). Now we consider the set

$$D = \big\{x \in v \in W_0^{1,p}(Z) : \underline{v}(z) \le v(z) \le 0 \ \text{ a.e. on } \ Z\big\} .$$

We introduce the truncation function $\tau_- : \mathbb{R} \to \mathbb{R}_-$. defined by

$$\tau_-(x) = \begin{cases} x & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases} .$$

Then $j_-(z, x) = j(z, \tau(x))$ and $\varphi_-(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z j_-(z, x(z))dz$ for all $x \in W_0^{1,p}(Z)$. We consider the minimization problem $\inf_D \varphi_-$. Reasoning as with φ_+ on C, we obtain:

Proposition 3.4. *If hypotheses $H(j)_1$ hold, then there exists $v_0 \in D$ which is a local minimizer of φ_- and of φ.*

Propositions 3.3 and 3.4, lead to the following multiplicity theorem for solutions of constant sign for problem (1.1).

Theorem 3.5. *If hypotheses $H(j)_1$ hold, then problem* (1.1) *has at least two constant sign smooth solutions $x_0 \in intC_+$ and $v_0 \in -intC_+$.*

Remark 3.6. Since x_0, v_0 are both local minimizers of φ, from the mountain pass theorem, we obtain a third critical point y_0 of φ, distinct from x_0, v_0. However, at this point we can not guarantee that $y_0 \neq 0$, let alone that it is nodal. This will be done in the next section under additional hypotheses.

4. Nodal solutions

In this section we produce a third nontrivial solution for problem (1.1) which is nodal (i.e., sign-changing). Our approach was inspired by the work of Dancer-Du [6]. Roughly speaking the strategy is the following: Continuing the argument employed in Section 3, we produce a smallest positive solution y_+ and a biggest negative solution y_-. In particular $\{y_\pm\}$ is an ordered pair of upper-lower solutions. So, if we form the order interval $[y_-, y_+]$ and we argue as in Section 3, we can show that problem (1.1) has a solution $y_0 \in [y_-, y_+]$ distinct from y_-, y_+. If we can show that $y_0 \neq 0$, then clearly y_0 is a nodal solution of (1.1). To show the nontriviality of y_0, we use Theorem 2.4 and (2.3).

We start implementing the strategy, by proving that the set of upper (resp. lower) solutions for problem (1.1), is downward (resp. upward) directed. The proof relies on the use of the truncation function

$$\xi_\varepsilon(s) = \begin{cases} -\varepsilon & \text{if } s < \varepsilon \\ s & \text{if } s \in [-\varepsilon, \varepsilon] \\ \varepsilon & \text{if } s > \varepsilon \end{cases} .$$

Note that

$$\frac{1}{\varepsilon}\xi_\varepsilon\big((y_1 - y_1)^-(z)\big) \to \chi_{\{y_1<y_2\}}(z) \quad \text{a.e. on } Z \text{ as } \quad \varepsilon \downarrow 0 .$$

So we have the following lemmata

Lemma 4.1. *If $y_1, y_2 \in W^{1,p}(Z)$ are two upper solutions for problem* (1.1) *and $y = \min\{y_1, y_2\} \in W^{1,p}(Z)$, then y is also an upper solution for problem* (1.1).

Lemma 4.2. *If $v_1, v_2 \in W^{1,p}(Z)$ are two lower solutions for problem* (1.1) *and $v = \max\{v_1, v_2\} \in W^{1,p}(Z)$, then v is also a lower solution for problem* (1.1).

In Section 3 we used zero as a lower solution for the "positive" problem and as an upper solution for the "negative" problem. However, this is not good enough for the purpose of generating a smallest positive and a biggest negative solution, as described earlier. For this reason, we strengthen the hypotheses on $j(z, x)$ as follows:

$\underline{H(j)_2}$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(t, 0) = 0$ a.e. on Z, $\partial j(z, 0) = \{0\}$ a.e. on Z, hypotheses $H(j)_2(i) \to (iv)$ and (vi) are the same as hypotheses $H(j)_1(i) \to (iv)$ and (vi) and

(iv) there exists $\widehat{\eta} \in L^\infty(Z)_+$, such that
$$\lambda_1 < \liminf_{x\to 0} \frac{u}{|x|^{p-2}x} \le \limsup_{x\to 0} \frac{u}{|x|^{p-2}x} \le \widehat{\eta}(z)$$
uniformly for a.a. $z \in Z$ and all $u \in \partial j(z,x)$.

Using this stronger hypothesis near origin, we can find $\mu_0 \in (0,1)$ small such that $\underline{x} = \mu_0 u_1 \in \mathrm{int}C_+$ is a strict lower solution and $\overline{v} = \mu_0(-u_1) \in -\mathrm{int}C_+$ is a strict upper solution for problem (1.1). So we can state the following lemma:

Lemma 4.3. *If hypotheses $H(j)_2$ hold, then problem* (1.1) *has a strict lower solution $\underline{x} \in intC_+$ and a strict upper solution $\overline{v} \in -intC_+$.*

We consider the order intervals
$$[\underline{x}, \overline{x}] = \left\{x \in W_0^{1,p}(Z) : \underline{x}(z) \le x(z) \le \overline{x}(z) \text{ a.e. on } Z\right\}$$
$$\text{and } [\underline{v}, \overline{v}] = \left\{v \in W_0^{1,p}(Z) : \underline{v}(z) \le v(z) \le \overline{v}(z) \text{ a.e. on } Z\right\}.$$
Using Lemmata 4.1 and 4.2 and Zorn's lemma, we prove the following result:

Proposition 4.4. *If hypotheses $H(j)_2$ hold, then problem* (1.1) *admits a smallest solution in the order interval $[\underline{x}, \overline{x}]$ and a biggest solution in the order interval $[\underline{v}, \overline{v}]$.*

Now let $\underline{x}_n = \varepsilon_n u_1$ with $\varepsilon_n \downarrow 0$ and let $E_+^n = [\underline{x}_n, \overline{x}]$. Proposition 4.4 implies that problem (1.1) has a smallest solution x_*^n in E_+^n. Clearly $\{x_*^n\}_{n\ge1} \subseteq W_0^{1,p}(Z)$ is bounded and so by passing to a suitable subsequence if necessary, we may assume that
$$x_*^n \xrightarrow{w} y_+ \text{ in } W_0^{1,p}(Z) \quad \text{and} \quad x_*^n \to y_+ \text{ in } L^p(Z) \text{ as } n \to \infty.$$
Arguing by contradiction and using hypothesis $H(j)_2(v)$, we can show that $y_+ \ne 0$ and of course $y_+ \ge 0$. Here we use the strict monotonicity of the principal eigenvalue on the weight function (see Section 2). Moreover, by Vazquez [14], we have $y_+ \in \mathrm{int}C_+$ and using this fact it is not difficult to check that y_+ is in fact the smallest positive solution of problem (1.1).

Similarly, working on $E_-^n = [\underline{v}, \overline{v}_n]$ with $\overline{v}_n = \varepsilon_n(-u_1)$, $\varepsilon_n \downarrow 0$, we obtain $y_- \in -\mathrm{int}C_+$ the biggest negative solution of (1.1). So we can state the following proposition:

Proposition 4.5. *If hypotheses $H(j)_2$ hold, then problem* (1.1) *has a smallest positive solution $y_+ \in intC_+$ and a biggest negative solution $y_- \in -intC_+$.*

According to the scheme outlined in the beginning of the section, using this proposition, we can establish the existence of a nodal solution. As we already mentioned, a basic tool to this end, is equation (2.3). But in order to be able to use (2.3), we need to strengthen further our hypothesis near the origin. Also we need to restrict the kind of locally Lispchitz functions $j(z,x)$, we have. Namely, let $f : Z \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that for every $r > 0$ there exists $a_r \in L^\infty(Z)_+$ such that
$$|f(z,x)| \le a_r(z) \text{ for a.a. } z \in Z \quad \text{and all } |x| \le r.$$

We introduce the following two limit functions:

$$f_1(z,x) = \liminf_{x' \to x} f(z,x') \quad \text{and} \quad f_2(z,x) = \limsup_{x' \to x} f(z,x')\,.$$

Both functions are $\mathbb{R}$-valued for a.a. $z \in Z$. In addition we assume that they are sup-measurable, meaning that for every $x : Z \to \mathbb{R}$ measurable function, the functions $z \to f_1(z,x(z))$ and $z \to f_2(z,x(z))$ are both measurable. We set

$$j(z,x) = \int_0^x f(z,s)ds\,. \tag{4.1}$$

Evidently $(z,x) \to j(z,x)$ is jointly measurable and for a.a. $z \in Z$, $x \to j(z,x)$ is locally Lipschitz. We have

$$\partial j(z,x) = \big[f_1(z,x), f_2(z,x)\big] \quad \text{for a.a. } z \in Z\,, \quad \text{for all } x \in \mathbb{R}\,.$$

Clearly $j(z,0) = 0$ a.e. on Z and if for a.a. $z \in Z$, $f(z,\cdot)$ is continuous at 0, then $\partial j(z,0) = \{0\}$ for a.a. $z \in Z$. The hypotheses on this particular nonsmooth potential function $j(z,x)$ are the following:

$\underline{H(j)_3}$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is defined by (4.1) and

(i) $(z,x) \to f(z,x)$ is measurable with f_1, f_2 sup-measurable;
(ii) for a.a. $z \in Z$, $x \to f(z,x)$ is continuous at $x = 0$;
(iii) $|f(z,x)| \le a(z) + c|x|^{p-1}$ a.e. on Z, for all $x \in \mathbb{R}$,with $a \in L^\infty(Z)_+$, $c > 0$;
(iv) there exists $\theta \in L^\infty(Z)_+$ satisfying $\theta(z) \le \lambda_1$ a.e. on Z, $\theta \ne \lambda_1$ and

$$\limsup_{|x| \to \infty} \frac{f_2(z,x)}{|x|^{p-2}x} \le \theta(z)$$

uniformly for a.a. $z \in Z$;
(v) there exists $\widehat{\eta} \in L^\infty(Z)_+$ such that

$$\lambda_2 < \liminf_{x \to 0} \frac{f_1(z,x)}{|x|^{p-2}x} \limsup_{x \to 0} \frac{f_2(z,x)}{|x|^{p-2}x} \le \widehat{\eta}(z)$$

uniformly for a.a. $z \in Z$;
(vi) for a.a. $z \in Z$ and all $x \in \mathbb{R}$, we have $f_1(z,x)x \ge 0$ (sign condition).

From Proposition 4.5, we have a smallest positive solution $y_+ \in \mathrm{int}C_+$ and a biggest negative solution $y_- \in -\mathrm{int}C_+$ for problem (1.1). We have

$$A(y_\pm) = u_\pm \ \text{ with } \ u_\pm \in L^{p'}(Z),\ u_\pm(z) \in \partial j\big(z, x_\pm(z)\big) \quad \text{a.e. on } Z\,.$$

We introduce the following truncations of the functions $f(z,x)$:

$$\widehat{f}_+(z,x) = \begin{cases} 0 & \text{if } x<0 \\ f(z,x) & \text{if } 0 \le x \le y_+(z) \\ u_+(z) & \text{if } y_+(z) < x \end{cases},$$

$$\widehat{f}_-(z,x) = \begin{cases} u_-(z) & \text{if } x<y_-(z) \\ f(z,x) & \text{if } y_-(z) \le x \le 0 \\ 0 & \text{if } 0 < x \end{cases},$$

$$\widehat{f}(z,x) = \begin{cases} u_-(z) & \text{if } x<y_-(z) \\ f(z,x) & \text{if } y_-(z) \le x \le y_+(z) \\ u_+(z) & \text{if } y_+(z) < x \end{cases},$$

Using them, we define the corresponding locally Lipschitz potential functions, namely $\widehat{j}_+(z,x) = \int_0^x \widehat{f}_+(z,s)ds$, $\widehat{j}_-(z,x) = \int_0^x \widehat{f}_-(z,s)ds$ and $\widehat{j}(z,x) = \int_0^x \widehat{f}(z,s)ds$ for all $(z,x) \in Z \times \mathbb{R}$.

Also, we introduce the corresponding locally Lipschitz Euler functionals defined on $W_0^{1,p}(Z)$. So we have

$$\widehat{\varphi}_+(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \widehat{j}_+\big(z,x(z)\big)dz,\ \widehat{\varphi}_-(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \widehat{j}_-\big(z,x(z)\big)dz$$

$$\text{and } \ \widehat{\varphi}(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \widehat{j}\big(z,x(z)\big)dz \ \text{ for all } x \in W_0^{1,p}(Z)\,.$$

Finally, we set

$$T_+ = [0,y_+],\ T_- = [y_-,0] \text{ and } T = [y_-,y_+]\,.$$

We can show that the critical points of φ_+ (resp. of φ_-, φ) are in T_+ (resp. in T_-,T). So the critical points of $\widehat{\varphi}_+$ (resp. $\widehat{\varphi}_-$) are $\{0,y_+\}$ (resp. $\{0,y_-\}$). Moreover,

$$\widehat{\varphi}_+(y_+) = \inf \widehat{\varphi}_+ < 0 = \widehat{\varphi}_+(0) \quad \text{and} \quad \widehat{\varphi}_-(y_-) = \inf \widehat{\varphi}_- < 0 = \widehat{\varphi}_-(0)\,.$$

Clearly y_+, y_- are local $C_0^1(\overline{Z})$-minimizers of $\widehat{\varphi}$ and so they are also local $W_0^{1,p}(Z)$-minimizers. Without any loss of generality, we may assume that they are isolated critical points of $\widehat{\varphi}$. So we can find $\delta > 0$ small such that

$$\widehat{\varphi}(y_-) < \inf\big[\widehat{\varphi}(x) : x \in \partial B_\delta(y_-)\big] \le 0\,,$$
$$\widehat{\varphi}(y_+) < \inf\big[\widehat{\varphi}(x) : x \in \partial B_\delta(y_+)\big] \le 0\,,$$

where $\partial B_\delta(y_\pm) = \{x \in W_0^{1,p}(Z) : \|x - y_\pm\| = \delta\}$. Assume without loss of generality that $\widehat{\varphi}(y_-) \le \widehat{\varphi}(y_+)$.

If we set $S = \partial B_\delta(y_+)$, $T_0 = \{y_-, y_+\}$ and $T = [y_-, y_+]$, then we can check that the pair $\{T_0, T\}$ is linking with S in $W_0^{1,p}(Z)$. So by virtue of Theorem 2.2, we can find $y_0 \in W_0^{1,p}(Z)$ a critical point of $\widehat{\varphi}$ such that

$$\widehat{\varphi}(y_\pm) < \widehat{\varphi}(y_0) = \inf_{\overline{\gamma}\in\Gamma} \max_{t\in[-1,1]} \widehat{\varphi}\big(\gamma(t)\big) \tag{4.2}$$

where $\overline{\Gamma} = \{\overline{\gamma} \in C([-1,1], W_0^{1,p}(Z)) : \overline{\gamma}(-1) = y_-, \overline{\gamma}(1) = y_+\}$. Note that from (4.2) we infer that $y_0 \ne y_\pm$.

We will show that $\widehat{\varphi}(y_0) < \widehat{\varphi}(0) = 0$ and so $y_0 \neq 0$. Hence y_0 is the desired nodal solution. To establish the nontriviality of y_0, it suffices to construct a path $\overline{\gamma}_0 \in \overline{\Gamma}$ such that

$$\widehat{\varphi}(\gamma_0(t)) < 0 \quad \text{for all } \ t \in [0,1] \ \text{ (see (4.2))}.$$

Using (2.3), we can produce a continuous path γ_0 joining$-\varepsilon u_1$ and εu_1 for $\varepsilon > 0$ small. Note that if $S_c = C_0^1(\overline{Z}) \cap \partial B_1^{L^p(Z)}$ and $S = W_0^{1,p}(Z) \cap \partial B_1^{L^p(Z)}$ are equipped with the relative $C_0^1(\overline{Z})$ and $W_0^{1,p}(Z)$ topologies respectively, then

$$C\big([-1,1], S_c\big) \quad \text{is dense in } \ C\big([-1,1], S\big).$$

Also we have

$$\widehat{\varphi}|_{\gamma_0} < 0. \tag{4.3}$$

Using Theorem 2.4, we can generate the continuous path

$$\gamma_+(t) = h(t, \varepsilon u_1), \quad t \in [0,1],$$

with $h(t,x)$ the deformation of Theorem 2.4. This path joins εu_1 and y_+. Moreover, we have

$$\widehat{\varphi}|_{\gamma_+} < 0. \tag{4.4}$$

In a similar fashion we produce a continuous path γ_- joining y_- with $-\varepsilon u_1$ such that

$$\widehat{\varphi}|_{\gamma_-} < 0. \tag{4.5}$$

Concatinating γ_-, γ_0 and γ_+, we produce a path $\overline{\gamma}_0 \in \overline{\Gamma}$ such that

$$\widehat{\varphi}|_{\overline{\gamma}_0} < 0 \ \text{ (see (4.3),(4.4) and (4.5))}.$$

This proves that $y_0 \neq 0$ and so y_0 is a nodal solution. Nonlinear regularity theory implies that $y_0 \in C_0^1(\overline{Z})$.

Therefore we can state the following theorem on the existence of nodal solutions

Theorem 4.6. *If hypotheses $H(j)_3$ hold, then problem* (1.1) *has a nodal solution* $y_0 \in C_0^1(\overline{Z})$.

Combining Theorems 3.5 and 4.6, we can state the following multiplicity result for problem (1.1).

Theorem 4.7. *If hypotheses $H(j)_3$ hold, then problem* (1.1) *has at least three nontrivial solutions, one positive $x_0 \in intC_+$, one negative $v_0 \in -intC_+$ and the third $y_0 \in C_0^1(\overline{Z})$ nodal.*

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